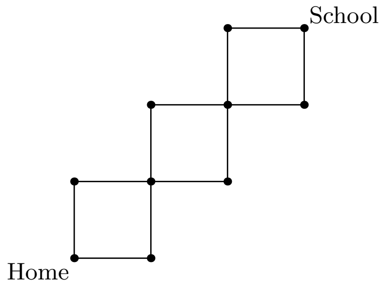
2022 Team Round Solutions

Key (solutions start on next page)

1. 8
2. 14110
3. 1
4. 4
5. 30
6. 13
7. 3
8. 5
9. 259
10. 61

Solutions

1. Alex is traveling from home to school along the roads shown below. Every minute, he either walks a segment up or to the right, until he arrives at school 6 minutes later. How many possible paths are there for Alex to go to school?



Answer: $\boxed{8}$.

Solution: Alex has 2 ways to decide how to walk for the first two minutes. Then he has 2 ways to decide on how to walk for the next two minutes. Finally, he has 2 ways to decide how to walk for the final 2 minutes. This means that in total, there are $2 \cdot 2 \cdot 2 = 8$ ways for Alex to go to school. □

2. The letters N, C, S, M each represent a digit from $0 - 9$ such that

$$\begin{array}{cccc} & & \text{N} & \text{C} \\ & \text{S} & \text{M} & \text{C} \\ + & \text{S} & \text{M} & \text{C} \\ \hline & 2 & 2 & 2 \end{array}$$

The digits N and S are not zero. What is the 5 digit number NCSSM?

Answer: $\boxed{14110}$.

Solution: We want $C + C + C$ to end with units digit 2. The only value of C that satisfies this is $C = 4$. In addition, we see that we must have $S = 1$ (it cannot be any larger). In order for the sum to work out, the remaining condition is that $N + M + M = 1$. This means we must have $N = 1, M = 0$. Extracting the answer, NCSSM is the 5 digit number 14110. □

3. One of four siblings, Barry, Harry, Larry and Mary, stole a cookie from the cookie jar. Three of these siblings always lie, and one of them always tells the truth. When asked about who stole the cookie from the cookie jar, they gave the following responses:

- Barry: Larry stole the cookie from the cookie jar
- Harry: I stole the cookie from the cookie jar
- Larry: I did not steal the cookie from the cookie jar
- Mary: Barry did not steal the cookie from the cookie jar

Who stole the cookie from the cookie jar? Submit 1 if the answer is Barry, 2 if the answer is Harry, 3 if the answer is Larry, and 4 if the answer is Mary.

Answer: $\boxed{1}$.

Solution: One of Barry and Larry must be the truthteller. No matter which one it is, Mary must be a liar. This means Barry stole the cookie. □

4. How many different ways are there to write 6 as a sum of n consecutive integers for some positive integer n? For example, $1+2+3=6$ is a way to write 6 as a sum of 3 consecutive integers.

Answer: $\boxed{4}$.

Solution: We can check that there exactly four values of n that work:

- $n=12$ works: $(-5)+(-4)+(-3)+\cdots+5+6=6$
- $n=4$ works: $0+1+2+3=6$
- $n=3$ works: $1+2+3=6$
- $n=1$ works: $6=6$

□

5. How many ways are there to arrange the letters in the word POTATO such that all the O's come before all the T's?

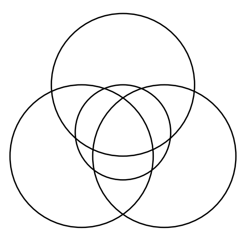
Answer: $\boxed{30}$.

Solution: Notice that it's equivalent to arranging the letters in the word PXXAXX. Wherever the X's end up, the first two must be the O's, and the second two must be T's. The number of ways to rearrange PXXAXX is $\frac{6!}{4!}=30$. □

6. Four circles are drawn in a plane, such that every circle intersects every other circle twice. How many regions of finite area do these four circles split the plane into?

Answer: $\boxed{13}$.

Solution: It's possible to just draw it out and count:



There are a total of 13 finite regions. □

7. The binary operation ♠ is defined as $a♠b=ab-3a-b+6$. For example, $4♠5=4\cdot5-3\cdot4-5+6=9$. What is the value of the below expression?

$$2022♠(2021♠(2020♠(\ldots(2♠(1♠0))\ldots)))$$

Answer: $\boxed{3}$.

Solution: Compute $1♠0=3$. Now, observe that no matter what the value of a is, we have $a♠3=3a-3a-3+6=3$. This means that the entire expression will end up equal to 3. □

8. If Sukrith, his enemy Rohan, and 6 others sit randomly in a line, what is the chance that they are sitting next to each other? If the probability is $\frac{a}{b}$ for relatively prime positive integers a and b, submit the number $a+b$.

Answer: $\boxed{5}$.

Solution: We split into two cases:

- If Sukrith is one of the two people on the end (this happens with probability $\frac{2}{8}$), there is a $\frac{1}{7}$ that his neighbor happens to be Rohan, out of the 7 possibilities.
- Otherwise, if Sukrith is in the middle 6 people (this happens with probability $\frac{6}{8}$), there is a $\frac{2}{7}$ chance that Rohan is one of Sukrith's two neighbors.

Overall, this gives a probability of $\frac{2}{8} \cdot \frac{1}{7} + \frac{6}{8} \cdot \frac{2}{7} = \frac{1}{4}$ that Sukrith and Rohan are neighbors. □

9. Isaac rolls three fair standard six-sided dice. The probability that there is at least one pair of dice whose top faces sum to at least 11 is $\frac{a}{b}$, where a and b are relatively prime positive integers. What is $a+b$?

Answer: $\boxed{259}$.

Solution: First count the number of ordered triples that work. We have two cases:

- There are two dice that sum to 12. Then the numbers on the dice must be $(6, 6, n)$ for some $n = 1, 2, 3, 4, 5, 6$. If $n \neq 6$, there are 3 ways to rearrange the numbers on the dice; on the other hand, $n = 6$ only corresponds to one possibility. Hence the total number of ways in this case is $3 \cdot 5 + 1 = 16$.
- There are two dice that sum to 11, but no two dice sum to 12. Then the numbers on the dice must be $(5, 6, n)$ for some $n = 1, 2, 3, 4, 5$. If $n = 5$ then there are 3 ways to rearrange the numbers on the dice; otherwise, there are 6 ways to rearrange the numbers on the dice. Hence the total number of ways in this case is $6 \cdot 4 + 3 = 27$.

In total, there are $27 + 16 = 43$ ordered triples that work. Out of $6^3 = 216$ possibilities, this corresponds to a probability of $\frac{43}{216}$. The requested answer is 259. □

10. Find the number of ordered triples of integers (a, b, c) with $1 \leq a, b, c \leq 10$ such that

- $a - b$ is a multiple of c (i.e. $\frac{a-b}{c}$ is a integer)
- $b - c$ is a multiple of a
- $c - a$ is a multiple of b

Answer: $\boxed{61}$.

Solution: Look at the greatest number out of a, b, c. Consider the case when this is a: we're given that $b - c$ is a multiple of a, but we also must have that $a > |b - c|$. Hence we must have $b - c = 0$, i.e. $b = c$.

This means that two of (a, b, c) must be equal, no matter what. It follows that the third number must be a multiple of the other two. So we wish to count the number of ordered triples of the form (k, k, nk), or its permutations.

If $k = 1$, we can count $1 + 9 \cdot 3$ possibilities. If $k = 2$, we can count $1 + 4 \cdot 3$ possibilities. If $k = 3$, we can count $1 + 2 \cdot 3$ possibilities. If $k = 4$, we can count $1 + 3$ possibilities. If $k = 5$, we can count $1 + 3$ possibilities. For all $k \geq 6$, there is only one possibility (we must have $n = 1$). Summing up, we find $28 + 13 + 7 + 4 + 4 + 5 = 61$ possibilities. □